18th Irish Mathematical Olympiad, 7 May 2005
First Paper: 10 a.m. – 1 p.m.

1. Prove that 2005^{2005} is a sum of two perfect squares, but not the sum of two perfect cubes.

2. Let ABC be a triangle and let D, E and F, respectively, be points on the sides BC, CA and AB, respectively—none of which coincides with a vertex of the triangle—such that AD, BE and CF meet at a point G. Suppose the triangles AGF, CGE and BGD have equal area. Prove that G is the centroid of ABC.

3. Prove that the sum of the lengths of the medians of a triangle is at least three quarters of the sum of the lengths of the sides.

4. Determine the number of different arrangements $a_1, a_2, \ldots, a_{10}$ of the integers $1, 2, \ldots, 10$ such that
$$a_i > a_{2i} \text{ for } 1 \leq i \leq 5,$$
and
$$a_i > a_{2i+1} \text{ for } 1 \leq i \leq 4.$$

5. Suppose a, b and c are non-negative real numbers. Prove that
$$\frac{1}{3}[(a-b)^2+(b-c)^2+(c-a)^2] \leq a^2+b^2+c^2-3\sqrt[3]{a^2b^2c^2} \leq (a-b)^2+(b-c)^2+(c-a)^2.$$

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Second Paper: 2 p.m. – 5 p.m.

1. Let ABC be a triangle, and let X be a point on the side AB that is not A or B. Let P be the incentre of the triangle ACX, Q the incentre of the triangle BCX and M the midpoint of the segment PQ. Show that $|MC| > |MX|$.

2. Using only the digits $1, 2, 3, 4$ and 5, two players A, B compose a 2005-digit number N by selecting one digit at a time as follows: A selects the first digit, B the second, A the third and so on, in that order. The last to play wins if and only if N is divisible by 9. Who will win if both players play as well as possible?

3. Suppose that x is an integer and y, z, w are odd integers. Show that 17 divides $x^{y^{z^w}} - x^{y^z}$.

4. Find the first digit to the left, and the first digit to the right, of the decimal point in the decimal expansion of $(\sqrt{2} + \sqrt{5})^{2000}$.

5. Let m, n be odd integers such that $m^2 - n^2 + 1$ divides $n^2 - 1$. Prove that $m^2 - n^2 + 1$ is a perfect square.